springer 1998.

W15. (**Solution by the proposer.**) The inequality is actually symmetric. For instance we can exchange a with b and get

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}-\left(\frac{b^2}{a+b}+\frac{a^2}{a+c}+\frac{c^2}{c+b}\right)=\sum_{\text{cyc}}\frac{a^2-b^2}{a+b}=\sum_{\text{cyc}}(a-b)=0$$

and

$$\sum_{\text{cyc}}\frac{a^3}{b^2+c^2}$$

is clearly symmetric.
Clearing the denominators in

$$\sum_{\text{cyc}}\frac{2a^2}{a+b}+\frac{(a^2+b^2+c^2)^2}{ab^2+ba^2+bc^2+cb^2+ca^2+ac^2}\geq\frac{9}{2}\frac{a^2+b^2+c^2}{a+b+c}$$

we obtain

$$\sum_{\text{sym}}2a^7b+3(ab)^4\geq\sum_{\text{sym}}(a^6b^2+a^6bc+2a^4b^2c^2+a^3b^3c^2)$$

and the conclusion follows by using the AGM.
Indeed

$$5a^7b+ab^7\geq 6a^6b^2\implies\sum_{\text{sym}}a^7b\geq\sum_{\text{sym}}a^6b^2$$

$$36a^7b+b^7c+6c^7a\geq 43a^6bc\implies\sum_{\text{sym}}a^7b\geq\sum_{\text{sym}}a^6bc$$

$$(ab)^4+(ac)^4\geq 2a^4(bc)^2\implies\sum_{\text{sym}}(ab)^4\geq\sum_{\text{sym}}a^4(bc)^2$$

$$2(ab)^4+(bc)^4+(ca)^4\geq 4(ab)^3c^2\implies\sum_{\text{sym}}(ab)^4\geq\sum_{\text{sym}}(ab)^3c^2$$

and the proof is complete.

Second solution. Noting that

$$\frac{2a^2}{a+b} = \frac{a^2+b^2}{a+b} + \frac{a^2-b^2}{a+b} = \frac{a^2+b^2}{a+b} + a - b$$

and $\sum\limits_{cyc}(a-b)=0$ we obtain

$$\sum_{cyc}\left(\frac{2a^2}{a+b}+\frac{a^3}{b^2+c^2}\right) \ge \frac{9}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c} \iff$$

$$\iff \sum_{cyc}\left(\frac{a^2+b^2}{a+b}+\frac{a^3}{b^2+c^2}\right) \ge \frac{9}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c} \iff$$

$$\iff \sum_{cyc}\left(\frac{\left(a^2+b^2\right)(a+b+c)}{a+b}+\frac{a^3(a+b+c)}{b^2+c^2}\right) \ge \frac{9}{2}\left(a^2+b^2+c^2\right) \iff$$

$$\iff \sum_{cyc}\frac{c\left(a^2+b^2\right)}{a+b}+\sum_{cyc}\left(a^2+b^2\right)+\sum_{cyc}\frac{a^4}{b^2+c^2}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2} \ge$$

$$\ge \frac{9}{2}\left(a^2+b^2+c^2\right) \iff$$

$$\iff \sum_{cyc}\frac{c\left(a^2+b^2\right)}{a+b}+2\left(a^2+b^2+c^2\right)+\sum_{cyc}\frac{a^4}{b^2+c^2}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2} \ge$$

$$\ge \frac{9}{2}\left(a^2+b^2+c^2\right) \iff$$

$$\iff \sum_{cyc}\frac{c\left(a^2+b^2\right)}{a+b}+\sum_{cyc}\frac{a^4}{b^2+c^2}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2} \ge \frac{5}{2}\left(a^2+b^2+c^2\right).$$

Since

$$\sum_{cyc}\frac{c\left(a^2+b^2\right)}{a+b}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2} = \sum_{cyc}\frac{a\left(b^2+c^2\right)}{b+c}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2} =$$

$$=\sum_{cyc} a\left(\frac{b^2+c^2}{b+c}+\frac{a^2(b+c)}{b^2+c^2}\right)\geq\sum_{cyc} a\cdot 2\sqrt{\frac{b^2+c^2}{b+c}\cdot\frac{a^2(b+c)}{b^2+c^2}}=2\sum_{cyc} a^2$$

and by Cauchy Inequality

$$\sum_{cyc}\frac{a^4}{b^2+c^2}\geq\frac{\left(a^2+b^2+c^2\right)^2}{\sum_{cyc}(b^2+c^2)}=\frac{a^2+b^2+c^2}{2}$$

then

$$\sum_{cyc}\frac{c\left(a^2+b^2\right)}{a+b}+\sum_{cyc}\frac{a^4}{b^2+c^2}+\sum_{cyc}\frac{a^3(b+c)}{b^2+c^2}\geq 2\left(a^2+b^2+c^2\right)+$$

$$+\frac{a^2+b^2+c^2}{2}=\frac{5}{2}\left(a^2+b^2+c^2\right).$$

Arkady Alt

W16. (Solution by the proposer.) The inequality is implied by

$$\sin t\left(\cos^3 t+\sin t\right)\geq\left(\frac{2t}{\pi}\right)^{\cos^2 t},\quad 0\leq t\leq\frac{\pi}{4}$$

By the weighted-AGM we have

$$\sin t\cos t\cdot\cos^2 t+1\cdot\sin^2 t\geq\left(\frac{1}{2}\sin(2t)\cdot 1\right)^{\cos^2 t}$$

Moreover by $\sin(2t)\geq(4t)/\pi$ for $0\leq 2t\leq\frac{\pi}{2}$ we get

$$\sin t\cos t\cdot\cos^2 t+1\cdot\sin^2 t\geq\left(\frac{1}{2}\sin(2t)\cdot 1\right)^{\cos^2 t}\geq\left(\frac{2t}{\pi}\right)^{\cos^2 t}$$

whence the inequality.

W17. (Solution by the proposer.) The case $p=1$ is trivial because the solution of the recurrence $a_{n+1}=\frac{a_n}{1+a_n}$ is exactly $a_n=\frac{1}{n}$ setting to zero the terms of the series.
If $p\neq 1$, the proof amounts to find the first two terms of the expansion of a_n. We observe that $a_{n+1}<a_n$ and $a_n>0$. It follows the existence of the limit